

REALIZING SIMPLY CONNECTED 4-MANIFOLDS BY BLOWING DOWN

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A USUAL STARTING point in studying a smooth, compact, simply connected 4-manifold W^4 is to describe W^4 as a handlebody. In the case where W^4 can be obtained by attaching 2-handles to B^4 , Kirby's calculus of framed links [1] is available as a means for studying W^4 . Many interesting manifolds admit such decompositions and Kirby's techniques have led to a deeper understanding of a number of these. Unfortunately, not every simply-connected W^4 admits a decomposition of this restricted type: A theorem of Casson (based on a group theoretic result of Gerstenhaber and Rothaus [2]) shows that for some simply connected 4-manifolds with boundary every handlebody structure contains 1-handles. In approaching the general simply-connected W^4 one must come to terms with the existence of 1- and 3-handles. Recent work of Trace [3] has clarified the role of the 3-handles: If two 4-manifolds are both obtained by attaching the same number of 3-handles to the same simply connected 4-manifold and both have connected boundary, the two must be diffeomorphic. This paper is concerned with simply-connected handlebodies obtained by attaching 1- and 2-handles to B^4 . It is shown that each of these which satisfies a possibly vacuous group theoretic condition can be obtained from B^4 by surgering ribbon disks in B^4 and then attaching 2-handles as warranted by the second Betti number. It follows (Corollary 2) that every smooth, compact, simply connected W^4 can be obtained by adding 2-, 3- and 4-handles to a manifold obtained from B^4 by surgery on ribbon disks. A further corollary (Corollary 1) states that if W is a smooth compact 4-manifold with $\pi_1(W) = H_3(W) = 0$, $W \# \#_{i=1}^n CP^2$ has a handle decomposition with no 1- or 3-handles for some n . The first portion of the paper is devoted to recalling definitions and facts which will be required later.

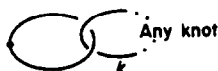
Suppose W^4 is a compact 4-manifold obtained by attaching 2-handles to the boundary of B^4 (a zero handle). Then W^4 can be described by a framed link in S^3 —the circles in ∂B^4 along which the handles are attached along with integers which fix the isotopy classes of the attaching maps. Kirby has devised a calculus of framed links for the study of such manifolds [1].

Suppose W^4 is a compact oriented 4-manifold obtained by attaching n 1-handles and m 2-handles to ∂B^4 . It is again possible to describe W^4 by a picture in S^3 . Since W^4 is oriented, the union of the zero and one-handles is the boundary connected sum of B^4 with n copies of $S^1 \times B^3$. The 2-handles are attached along framed curves in the resulting boundary, $S^3 \# \#_{i=1}^n (S^1 \times S^2)$. By general position we may assume that these curves lie in $S^3 \# \#_{i=1}^n (S^1 \times R^2)$ which is homeomorphic to the complement of a trivial link of n components in S^3 . Thus W^4 is completely described by a trivial link in S^3 together with a disjoint framed link.

Let L_1 be a (possibly empty) trivial link and L_2 a disjoint framed link in S^3 . Define the 4-manifold $W(L_1; L_2)$ to be obtained by attaching 1- and 2-handles to ∂B^4 as above. It is described by a picture of the link $L_1 \cup L_2$ in which each component of L_1 is marked with a

dot and each component of L_2 is accompanied by a framing. Suppose $D^2 \subset S^3$ is a disk spanning a component of L_1 and K is a component of L_2 . An intersection of K with D indicates that K is passing over the 1-handle represented by ∂D .

Example 1



The 4-manifold W described by this picture has one 1-handle and one 2-handle. Since the attaching curve of the 2-handle passes over the 1-handle geometrically once the handles cancel and $W \cong B^4$.

Suppose H is a finite oriented 4-dimensional handlebody with a single zero handle. Associated to H is a presentation P_H for the fundamental group of the manifold underlying H . (The generators and relators of P_H are fixed only up to conjugacy.) Since the union of the 0- and 1-handles is a boundary connected sum of B^4 with copies of $S^1 \times B^3$, each 1-handle of H gives rise to a generator in P_H realized by joining $S^1 \times \{\text{pt}\} \subset S^1 \times B^3$ to the basepoint. Each 2-handle of H gives rise to a relator in P_H obtained by joining its attaching circle to the basepoint and regarding the homotopy class (in $S^3 \#_{i=1}^n (S^1 \times S^2)$) of the resulting loop as a word in the generators.

Example 2



Our primary interest is in simply connected 4-manifolds described as handlebodies. However, in dealing with 4-dimensional handlebodies, simple connectivity is sometimes not a workable hypothesis—the fundamental group might die for reasons which are not apparent in the geometry of the handlebody. Similar difficulties arise in the study of contractible 2-complexes. In this context, Andrews and Curtis[4] have considered certain “moves” on group presentations which are geometrically realizable in the sense that an Andrews–Curtis move on P_H can be effected by a modification of H which does not change the underlying manifold. Listed below are the Andrews–Curtis moves and their geometric counterparts.

Let

$$\{x_1, \dots, x_n; r = 1, \dots, r_m = 1\}$$

be a group presentation.

Andrews–Curtis moves

- (i) Replace r_i by $g^{-1}r_i g$ where g is a word in the x_i .
- (ii) Replace r_i by $r_i r_j^\epsilon$ or $r_j^\epsilon r_i$ ($j \neq i$) $\epsilon = \pm 1$.
- (iii) Introduce a new generator x_{n+1} along with the relator $r_{m+1} = x_{n+1}$.

Geometric counterparts of the Andrews–Curtis moves

- (i)' Rechoose the path joining the attaching circle of a 2-handle to the basepoint.
- (ii)' Slide the attaching circle of a 2-handle over another 2-handle.
- (iii)' Introduce a new 1-handle and a 2-handle which cancels it geometrically.

Remark 1. The Andrews–Curtis moves together with (iv) add the new relator $r_{m+1} = 1$, are equivalent to the Tietze transformations. Thus, any two presentations for the same group are equivalent by moves of types (i)–(iv).

Remark 2. There exist presentations of isomorphic groups which are inequivalent by moves of types (i)–(iii) [5].

Remark 3. It is unknown whether there exist presentations of the trivial group which are not equivalent by moves of types (i)–(iii).

Definition. A presentation $\{x_1, \dots, x_n; r = 1, \dots, r_m = 1\}$ for the trivial group *slides away* if it can be transformed by moves of types (i)–(iii) to the presentation

$$\{x_1, \dots, x_s; x_1 = 1, \dots, x_s = 1, 1 = 1, \dots, 1 = 1\} \text{ for some } s \geq n.$$

If $K \subset S^3$ is a knot, K bounds an immersed 2-disk whose self-intersections are transverse and include no triple points or closed components. The self-intersections are then of two possible types.



Diagram 1.

Remark 4. Suppose $D \subset S^3$ is an immersed 2-disk. By “pushing a sheet of D across ∂D ” a closed component of self-intersection can be replaced by a ribbon and a ribbon by two clasps (see Diagram 2). Thus ∂D bounds an immersed disk with only clasp self-intersections.

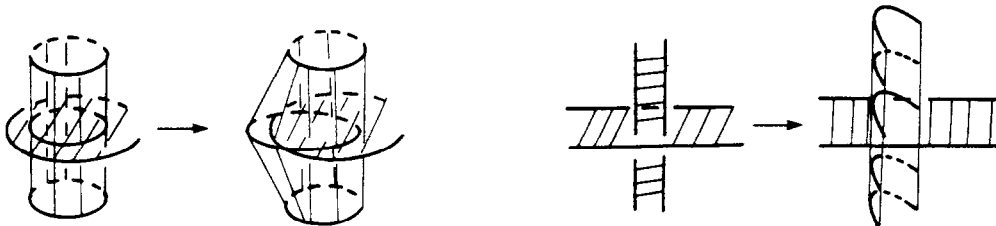


Diagram 2.

Definition. A collection $\{D_i\}_{i=1}^n$ of immersed 2-disks in S^3 is *ribbon* if each component of the self-intersection of $\bigcup_{i=1}^n D_i$ is a ribbon.

Remark 5. Suppose $\{D_i\}_{i=1}^n$ is a ribbon collection. There exist disjoint properly embedded 2-disks $\hat{D}_i \subset S^3 \times [0, 1] \subset S^3 \times [0, \infty) = B^4 - \{0\}$ such that (i) $\partial \hat{D}_i = \partial D_i$ and (ii) $p(\hat{D}_i) = D_i$ where $p: S^3 \times [0, 1] \rightarrow S^3$ is projection. Such a collection is unique up to isotopy and we denote by $\{\hat{D}_i\}_{i=1}^n$ any collection satisfying (i) and (ii).

Suppose $W(L_1, L_2)$ is a 4-manifold and $\{D_i\}_{i=1}^n$ a ribbon collection disjoint from L_1 such that $\{\partial D_i\}_{i=1}^n$ forms a sublink of L_2 . Associated to each D_i is a smooth 2-sphere $S(D_i) \subset W(L_1; L_2)$ defined as the union of \hat{D}_i with the core of the handle attached along ∂D_i . Since the \hat{D}_i are disjoint, so are the $S(D_i)$.

Suppose in addition that the framing associated to ∂D_i is ± 1 for $1 \leq i \leq n$. Then a tubular neighborhood of $S(D_i)$ is diffeomorphic with the Hopf disk bundle which has S^3 as boundary. The manifold $W(L_1; L_2)/\{S(D_i)\}_{i=1}^n$ is obtained from $W(L_1; L_2)$ by

deleting disjoint open tubular neighborhoods of the $S(D_i)$ and sewing in copies of B^4 along the resulting boundary components. This operation is known as “blowing down the $S(D_i)$.”

Example 3. Suppose the D_i are disjoint embedded disks which in addition are disjoint from the other components of L_2 . In this case blowing down the $S(D_i)$ is tantamount to move 1 of Kirby’s calculus of framed links [1]. Thus $W(L_1; L_2)$ is diffeomorphic to $W(L_1; L_2 - \{\partial D_i\}_{i=1}^n \# \#_{i=1}^n CP^2)$ and $W(L_1; L_2)/\{S(D_i)\}_{i=1}^n$ is diffeomorphic to $W(L_1; L_2 - \{\partial D_i\}_{i=1}^n)$.

Remark 6. Here is an alternate description of $W(L_1; L_2)/\{S(D_i)\}_{i=1}^n$ which does not involve blowing down. Each \hat{D}_i has a tubular neighborhood in $W(L_1; L_2 - \{\partial D_i\})$ diffeomorphic to $\hat{D}_i \times D^2$ with boundary $\hat{D}_i \times S^1$. Remove such a neighborhood of each \hat{D}_i and sew it back by the “twist” $H: D^2 \times S^1 \rightarrow D^2 \times S^1$ defined by $H(re^{i\theta}, e^{i\lambda}) = (re^{i\theta}, e^{i(\lambda - \theta)})$. (This sewing amounts to attaching handles with framing -1 .) We leave it to the reader to verify that the resulting manifold is $W(L_1; L_2)/\{S(D_i)\}_{i=1}^n$.

Remark 7. Melvin [6] has constructed an explicit handle presentation H for each $W(\phi; \{\partial D_k + 1\})/\{S(D_i)\}_{i=1}^n$ such that H has a single zero handle and no 3- or 4-handles and such that P_H slides away. Here is a sketch of Melvin’s proof: Place $\cup_{i=1}^n \hat{D}_i$ in critical level position and let $D^2 \times (\cup_{i=1}^n \hat{D}_i)$ be a tubular neighborhood. Let $J \subset \partial D^2$ be an arc. The key step is to find a handle structure K for $B^4 - (D^2 \times \text{int}(\cup_{i=1}^n \hat{D}_i))$. A small neighborhood of the origin is the 0-handle of K . Let $\{v\}$ be the core of a zero handle of $\cup_{i=1}^n \hat{D}_i$. By adding a collar to the 0-handle of K we may assume that $J \times \{v\}$ lies in its boundary (and that $(\partial D^2 - J) \times \{v\}$ remains outside the 0-handle). Then $(\partial D^2 - J) \times \{v\}$ is the core of a typical 1-handle of K . Let l be the core of a 1-handle of K . After adding a collar we may assume that $J \times l$ lies in the boundary of the union of the 0- and 1-handles of K . The disk $(\partial D^2 - J) \times l$ is the core of a typical 2-handle of K . Note that $\partial(\partial D^2 - J \times l)$ consists of four arcs: two of them pass geometrically once over a 1-handle and the other two are mutually parallel. Thus the relations of P_K are of the form $\{x_i g \bar{x}_j \bar{g} = 1\}$ where x_i and \bar{x}_j are generators. Since $\{D_i\}_{i=1}^n$ is ribbon, $\cup_{i=1}^n \hat{D}_i$ has no 2-handles and $B^4 - (D^2 \times \text{int}(\cup_{i=1}^n \hat{D}_i))$ no 3-handles. By Remark 6, H is obtained from K by attaching 2-handles with framing -1 along the n curves $(D^2 \times \{pt\}_i)$ where $\{pt\}_i$ is a point of \hat{D}_i . The relations in P_H arising from these 2-handles are of the form $\{x_i = 1\}$. It is now trivial to show that P_H slides away. Our main theorem is essentially the converse of Melvin’s result.

THEOREM. Suppose W^4 is a smooth, compact 4-manifold which admits a handle decomposition H such that: (i). H has a single zero handle and no handles of index 3 or 4, and (ii). P_H slides away. Then there exists a ribbon collection $\{D_i\}_{i=1}^n$ such that

$$W^4 \cong W(\phi; \{\partial D_i + 1\})/\{S(D_i)\}_{i=1}^n \cup \beta_2(W) \quad 2\text{-handles.}$$

Proof. Since P_H slides away, after geometrically realizing the Andrews–Curtis moves we may assume that $P_H = \{x_1, \dots, x_n; x_1 = 1, \dots, x_n = 1, 1 = 1, \dots, 1 = 1\}$. Since the conclusion allows attaching 2-handles it suffices to consider the case when $P_H = \{x_1, \dots, x_n; x_1 = 1, \dots, x_n = 1\}$ (i.e. $\beta_2(W) = 0$). Let $W = W(L_1; L_2)$ be a description of H . Let $\{J_i\}_{i=1}^n$ be the components of L_1 and $\{K_i\}_{i=1}^n$ the components of L_2 . Also define K'_i to be a small meridional circle about J_i for $i = 1, \dots, n$.

By the definition of P_H , K_i is homotopic to K'_i in $S^3 - L_1$. Thus there exist immersed annuli $A_i \subset S^3 - L_1$ with $\partial A_i = K_i \cup K'_i$. We may assume that the self-intersections of $\cup_{i=1}^n A_i$ are transverse and disjoint from $\cup_{i=1}^n K'_i$ (this because K'_i is close to J_i). After pushing sheets of $\cup_{i=1}^n A_i$ across $\cup_{i=1}^n K_i$ we may assume that each

component of intersection is a clasp near $\cup_{i=1}^n K_i$ (see Remark 4). Note that such a clasp can be eliminated by changing a crossing of the strands of $\cup_{i=1}^n K_i$ near the clasp. Thus there exist curves \bar{K}_i obtained from the K_i by changing crossings and disjointly embedded annuli $\bar{A}_i \subset S^3 - L_1$ with $\partial \bar{A}_i = \bar{K}_i \cup K'_i$ for $1 \leq i \leq n$.

In order to replace the K_i with the \bar{K}_i we will stabilize W by blowing up. It is shown in [7] that a crossing of two strands in $S^3 = \partial B^4$ can be changed in $S^3 = \partial(CP^2 - \bar{B}^4) = \partial(B^4 \cup (2\text{-handle}))$ by sliding the strands over the new 2-handle. The 2-handle is attached with framing +1 along the boundary of a 2-disk near the crossing but disjoint from its strands. Choose a 2-disk near each component of self-intersection of $\cup_{i=1}^n A_i$ and label these disks D_j $j = 1, \dots, m$. Attach handles with framing +1 along the ∂D_j . Let $F: W(L_1, L_2) \# \#_{i=1}^m (CP^2)_i \rightarrow W(L_1; L_2 \cup \{\partial D_j, +1\})$ be a diffeomorphism which sends the generator $(CP^1)_j \subset (CP^2)_j$ to $S(D_j)$ for $j = 1, \dots, m$ (see Example 3).

Perform handle slides to replace the K_i with framed curves \bar{K}_i such that there exist disjoint, embedded annuli $\bar{A}_i \subset S^3 - L_1$ with $\partial \bar{A}_i = \bar{K}_i \cup K'_i$. These slides induce a diffeomorphism $G: W(L_1; L_2 \cup \{\partial D_j, +1\}) \rightarrow W(L_1; \{\bar{K}_i\} \cup \{\partial D_j, +1\})$ such that $G(S(D_j)) = S(D_j)$ $j = 1, \dots, m$.

Since the \bar{A}_i are disjoint and embedded in $S^3 - L_1$ there exists an isotopy $h_i: S^3 \rightarrow S^3$ which slides \bar{K}_i to K'_i , $1 \leq i \leq n$, and which fixes a neighborhood of L_1 . The isotopy induces a diffeomorphism $H: W(L_1; \{\bar{K}_i\} \cup \{\partial D_j, +1\}) \rightarrow W(L_1; \{K'_i\} \cup \{h_1(\partial D_j), +1\})$. By choosing the diffeomorphism to restrict to h_1 on each slice of $S^3 \times [0, 1] \subset S^3 \times [0, \infty] = B^4 - \{0\}$ we can arrange that $H(S(D_j)) = S(h_1(\partial D_j))$ $j = 1, \dots, m$.

Note that the 2-handles attached along the K'_i geometrically cancel the 1-handles. The final step of the proof is to perform additional slides to picture the results of the cancellation.

Since L_1 is a trivial link there exist disjoint disks $\{E_i\}_{i=1}^n$ spanning its components. We may assume that the E_i intersect $\cup_{j=1}^m h_1(D_j)$ transversely and that $E_i \cap (\cup_{k=1}^n K'_k) = E_i \cap K'_i = \{pt.\}$. Since $L_1 \cap (\cup_{j=1}^m D_j) = \emptyset$ and h_1 fixes L_1 each component of $E \cap (\cup_{j=1}^m h_1(D_j))$ is contained in $\text{int}(E_i)$. After pushing sheets of E_i across ∂D_j we may assume that no components of intersection are closed (Remark 4). Each component must then be a ribbon.

Slide each component of $E_i \cap D_j$ along E_i and over the handle attached along K'_i . This move eliminates the component of $E_i \cap D_j$ (and slides the handle attached along ∂D_j over the handle attached along K'_i in two places). Let $j_i: \partial W \rightarrow \partial W$ be an isotopy of the identity which effects such a move on each component of $(\cup_{i=1}^n E_i) \cap (\cup_{j=1}^m h_1(D_j))$. Use a collaring of $\partial W \subset W$ to define a diffeomorphism $J: W(L_1; \{K'_i\} \cup \{h_1(\partial D_j), +1\}) \rightarrow W(L_1; \{K'_i\} \cup \{j_i h_1(\partial D_j), +1\})$ which restricts to j_i on ∂W .

Note that the support of J can be chosen to lie in an arbitrarily small neighborhood of $\cup_{i=1}^n (E_i \cup \text{handle over } K'_i)$. Thus we may assume that $\cup_{j=1}^m h_1(D_j)$ intersects $\text{supp}(J)$ only near the (ribbon) components of $(\cup_{j=1}^m h_1(D_j)) \cap (\cup_{i=1}^n E_i)$. After further adjusting the collaring we may assume that J restricts to $j_i \times \text{id}$ on the portion of $S^3 \times [0, \infty] = B^4 - \{0\}$ near $\cup_{j=1}^m h_1(D_j) \cap \text{supp}(J)$.

We wish to show that the spheres $\{J(S(h_1(D_j)))\}_{j=1}^m$ derive from a ribbon collection of disks. Clearly $J(S(h_1(D_j))) = (\text{core of handle over } j_i h_1(\partial D_j)) \cup J(\widehat{h_1(D_j)})$. It suffices to show that $\{p(J(\widehat{h_1(D_j)}))\}_{j=1}^m$ form a ribbon collection where $p: B^4 - \{0\} \rightarrow S^3$ is radial projection. Consider

$$\begin{aligned} p \circ J(\cup_{j=1}^m \widehat{h_1(D_j)}) &= p \circ J(\cup_{j=1}^m \widehat{h_1(D_j)} \cap \text{supp } J) \cup p \circ J(\cup_{j=1}^m \widehat{h_1(D_j)} - \text{supp } J) \\ &= p \circ (j_i \times \text{id})(\cup_{j=1}^m \widehat{h_1(D_j)} \cap \text{supp } J) \cup p(\cup_{j=1}^m \widehat{h_1(D_j)} - \text{supp } J) \\ &= j_i(\cup_{j=1}^m h_1(D_j) \cap \text{supp } J) \cup (\cup_{j=1}^m h_1(D_j) - \text{supp } J). \end{aligned}$$

The first piece of the union is a union of disks disjointly embedded in $\text{supp}(J)$. Since $\partial(\cup_{j=1}^m h_1(D_j) - \text{supp } J)$ is disjoint from $\text{supp } (J)$ each self-intersection of $p \circ J(\cup_{j=1}^m h_1(D_j))$ must be a ribbon.

Let $K: W(L_1; \{K_j\} \cup \{j_1 h_1(\partial D_j, +1)\}) \rightarrow W(\phi; \{j_1 h_1(\partial D_j, +1)\})$ be the diffeomorphism induced by handle cancellation. Choose the support of K to be close enough to $\cup_{j=1}^m (E_j \cup \text{handle over } K_j')$ to miss the spheres $\{J(S(h_1(D_j)))\}_{j=1}^m$.

Consider the composition $K \circ J \circ H \circ G \circ F: W(L_1; L_2) \# \#_{j=1}^m (CP^2)_j \rightarrow W(\phi; \{j_1 h_1(\partial D_j, +1)\})$. By construction $K \circ J \circ H \circ G \circ F[(CP^1)_j] = K \circ J(S(h_1(D_j))) = J(S(h_1(D_j)))$, $j = 1, \dots, m$ and these spheres derive from a ribbon collection of disks which have the $j_1 h_1(\partial D_j)$ as boundary. Recall that the manifold obtained by blowing down the $(CP^1)_j$ in $W(L_1, L_2) \# \#_{j=1}^m (CP^2)_j$ is $W(L_1; L_2)$. Thus blowing down the $(CP^1)_j$ and their images under the composition provides the desired diffeomorphism.

The first corollary is a stable result about handlebody structures on a compact simply connected 4-manifolds W with $H_3(W) = 0$.

COROLLARY 1. *Suppose W^4 is a smooth, compact, simply connected 4-manifold with $H_3(W) = 0$. Then $W \# \#_{i=1}^n CP^2$ admits a handlebody structure with no 1- or 3-handles for some integer n .*

Remark 8. Much stronger results due to Wall[8] are known for $W \# \#_{i=1}^n (S^2 \times S^2)$. However, it has been conjectured that the corresponding results fail for $W \# \#_{i=1}^n CP^2$.

Proof. Choose a handlebody structure for W which has a single zero handle and a single 4-handle (only in the case when W is closed). Let H be the handlebody consisting of the 0-, 1- and 2-handles of W . Note that the effect on P_H of adding CP^2 is to introduce a trivial relation $1 = 1$. By Remark 1, after adding copies of CP^2 we may assume that P_H slides away. The first part of the proof of the theorem shows that some manifold obtained by adding additional copies of CP^2 has a handle decomposition with no 1-handles. Thus $W' = W \# \#_{i=1}^k CP^2$ has a handle decomposition with no 1-handles.

A theorem of Trace[9] now shows that W' admits a handle decomposition with 1- and 2-handles, but no 3-handles. Repeating the above argument with this handlebody structure on W' in the place of H completes the proof.

The next corollary is an improvement on a result of Freedman[10]. Freedman has shown that every compact simply connected 4-manifold can be obtained by attaching 2-, 3- and 4-handles to a contractible 4-manifold. The corollary asserts that such a contractible 4-manifold can be obtained by blowing down ribbons. Aside from an application of the theorem our proof parallels Freedman's proof.

COROLLARY 2. *Suppose W^4 is a smooth compact, simply connected 4-manifold. Then there exists a ribbon collection $\{D_i\}_{i=1}^n$ such that W can be obtained by attaching 2-, 3- and 4-handles to $W(\phi; \{\partial D_i, +1\}_{i=1}^n) / \{S(D_i)\}_{i=1}^n$.*

Proof. Let H be a handlebody structure for W . Note that the effect on P_H of attaching a cancelling pair of 2- and 3-handles is to introduce a trivial relation $\{1 = 1\}$. By Remark 1 after attaching cancelling pairs we may assume that P_H slides away. An application of the theorem completes the proof.

The final corollary is included chiefly as a curiosity.

COROLLARY 3. *Suppose $\{D_i\}_{i=1}^n$ is a ribbon collection. There exists a ribbon collection $\{D'_j\}_{j=1}^m$ such that*

$$W(\phi; \{\partial D_i, -1\})/\{S(D_i)\}_{i=1}^n \cong W(\phi; \{\partial D'_j, +1\})/\{S(D'_j)\}_{j=1}^m$$

Proof. According to Melvin[6] (see Remark 7), $W(\phi; \{\partial D_i, -1\})/\{S(D_i)\}$ admits a handlebody structure H containing a single zero handle and with no handles of index 3 or 4 such that P_H slides away. An application of the theorem completes the proof.

REFERENCES

1. R. C. KIRBY: A calculus for framed links in S^3 . *Inventiones Math.* **45** (1978), 35–55.
2. M. GERTENHABER and O. S. ROTHUS: The solution sets of equations in groups. *PNAS* **48** (1962), 1531–1533.
3. B. TRACE: (to appear).
4. J. J. ANDREWS and M. L. CURTIS: Free groups and handlebodies. *PAMS* **16** (1965), 192–195.
5. M. J. DUNWOODY: Relation modules. *BLMS* **4** (1972), 151–155.
6. P. MELVIN: (unpublished).
7. S. J. KAPLAN: Constructing framed 4-manifolds with given almost framed boundaries. to appear in *TAMS*.
8. C. T. C. WALL: On simply connected 4-manifolds. *JLMS* **39** (1964), 141–149.
9. B. TRACE: A class of 4-manifolds which have 2-spines. *PAMS* **79** (1980), 155–156.
10. M. H. FREEDMAN: Cancelling 1-handles and some topological embeddings. *Pac. J.* **80** (1979), 127–130.

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